

7ci fgY[·]Cj Yfj]Yk

GHJZZ:7cbHJWh:8YHJ]`g`

7cbj Ybcfg

B Ua Y	9a U]`	5j U]`UV]`] m	@cWUh]cb	D\cbY
>cbUh\Ub`6c``Yb	G	A cbXUmgž ‰Ua!‰da /ch\Yf h]a Yg`Ya U]`Zcf	‰	

7ci fgY 8YHJ] g

7 fYX]h'Dc]bhg'*

Gi a a UfmcZh\Y 7ci fgY

: fca `XUbW]b[`fcVchg`hc`ga Ufhd\cbY`h\YUhfYgž'h\]g`Wci fgY`YI Ua]bYg`h\Y`g\]Zh]b[`fY`Uh]cbg\]d`VYkk YYb hYW\bc`c[mUbX`dYfZcfa UbW'`W`hi fYg"'±hWcbg]XYfg`h\Y`YI dUbX]b[`Z]Y`X`cZX][]hU``W`hi fY`UbX`\ck `]hg Udd`]WUh]cbg`\Uj Y`hfUbgZcfa YX`h\Y`dUfUa YhYfg`cZdYfZcfa UbW'`UbX`fYdfYgYbhUh]cb"`Mci `k]```cVgYfj Y`U fUb[Y`cZdYfZcfa UbW'`cV`YWgž`UWfj]hjYgž'Yj Ybhg`UbX`VY\Uj]ci fg`dfcXi WYX`]b`h\Y`YI W\Ub[Y`VYkk YYb`h\Y VcXmUbX`bYk 'a YX]U"`Mci `k]```XfUk `cb``YUX]b[`YI Ua d`Yg`hc`XYj Y`cd`mci f`ck b``]j Y`YI dYf]a Ybhg`UWfcgg d`UhZcfa g`h\Uhi gY`Hk]hhYfž'a cV]`Y`a YX]Už'gi fj Y]``UbW'`UbX`Mci Hi VY"`Mci `k]```YI d`cfY`h\Y`Wcbj Yf[YbW'`cZ dYfZcfa UbW'`[YbfYgž'h\Y`fYa YX]Uh]cb`cZUfhk cf_gž'h\YcfYh]WD``WcbW'dhg`gi W\`Ug`dcgh\i a Ub]ga `UbX WhYfbYh]Wgž'UbX`\ck ``]j Y`dYfZcfa UbW'`Yb[U[Yg`k]h\`WcbhYa dcfUfmWi`hi fU``W\Ub[Y"

7ci fgY @YUfb]b[Ci hWca Yg

- %" ⇒ YbhjZmUbX`X]gW' gg`_YmXYj Y`cda Ybhg`]b`WcbhYa dcfUfmidYfZcfa UbWY`]bgd]fYX`VmXYj Y`cda Ybhg`]b`hYW\bc`c[m
- &" 9 j U`i UhY h\Y'Yj c`j]b['fY`Uh|cbg\]d 'VYhk YYb hYW\bc`c[mUbX'UYgh\Yh]Wq
- ' " 5 ggYgg'h\Y'fc`Y'cZhYW\bc`c[m]b'dYfZcfa UbW\'\]ghcfm'UbX`]b'fYW\bh'h\YcfYh]WU`XYVUhYg']b'h\Y \i a Ub]h]Yg
- ("8Ya cbglfUhY`Ub`UXj UbWYX`_bck`YX[Y`cZX]ZZYfYbha cXYg`cZdYfZcfa UbWY`UbX`Ub`UddfYW]Uh]cb`cZh\Y]f`W``h fU``UbX`\]ghcf]WU``gdYW]ZW]m
-) " 6YHYfi bXYftu_Y`Wf]h]WU`XYVUHY`cb`WcbHYa dcfUfm'W`hi fU`]ggi Yq
- *"DifgiY]bX]j]XiU`[i]XYXfYgYUfW
- +" 7 cbg/fi Whgi ghJ]bYX Wf]h]WJ`Uf[i a Ybh]b h\Y Zcfa cZfYj]Yk g cf YggUmg
- , "8YacbghfuhYuhkcfci[_bckYX[YcZgYYWYXhkYcfYh]WuhdYfgdYWh]jYgubXfYgYufWkaYhkcXg

HYUWX]b[GhfUhY[]Yg

F Uh]cbU`Y

H\]g`Wci fgY`]g`XYg][bYX`hc`ZJV\]]hUhY`ghi XYbhg'``YUfb]b['VmYghUV`]g\]b['U`a i `h]a cXU``hYUW\]b['ghfi Whi fY`]b k\]W\ ghi XYbhg`X]gW ggžfYgYUfV\\ždfYgYbhž'dUff]\\]dUhY`UbX``]ghYb"`H\Y`cj YfU```U]a 'cZh\Y`Wci fgY`]g`hc YbUV`Y`ghi XYbhg`hc 'a U]bhU]b`U`XYYdYf`i bXYfghUbX]b['cZa YX]Uh]gYX'dYfZcfa UbWY`h\fci [\`ZJ\\]]hUh]b[WcbbY\\]cbg`h\Yma U_Y`VYhk YYb`fYUX]b[gž`\UgY`ghi X]Ygž'dfYgYbhUh]cbg`UbX`g\\YYb]b[g"`5ggYgga Ybhg fYZY\\hi\Y`YI dY\\Uh\]cbg`Zcf`YI \W``Yb\\Y`]b`fYgYUf\\\`UbX`]bj Ybhjj YbYgg`h\Uhi\]g`\\ci fgY`gYY_g`hc`YghUv`]g\" H\Y`hcd]\\g`h\Y`\\ci fgY`\\ci Yfg`fYZY\\hi\D`cb[c]b[`\\ci a a]ha Ybhhc`hYU\\]b[`]bbcj Uhjj YžfY`Yj UbhiUbX g][b]Z\\Ubha UhYf]\U`g`Zcf`h\Y`X]g\\]d\]bY`cZh\YUhfY#dYfZcfa Ub\W`ghi X]Yg

HYUWX]b['ghfUhY[]Yg

 G9A #B5F: CFA5H. H\Y'gYa]bUf'k]``VY``ccgY`mglfi Wi fYX'Ug.``YWi fY`Z'j]gi U``gWfYYb]b[gžZc``ck YX`Vmghi XYbhX]gW gg]cb'UbX'fYUX]b[g'fU'h\ci [\'h\]g'a UmW\Ub[Y'Zfca'k YY_hc'k YY_Ł"'H\Y'U]a 'cZh\]g'WUgg Zcfa Uh]g'hc']bhY[fUhY'U'bi a VYf'cZghi XYbh`YUfb]b[a cXU`]hJYg"'H\Y'`YWi fY']bZcfa Uh]cb'dfcj]XYg'U d`UhZcfa 'Zcf'ghi XYbhg'hc'h\Yb'Yb[U[Y']b'ghi XYbh'YX``YUfb]b['dfcWYggYg'h\fci [\'h\Y'U``cWUh]cb'cZZcW gei Ygh]cbg'UbX'#cf'gYhX]gW gg]cb'hUg_g"'Ghi XYbhg'UfY'UV'Y'hc'fYZYWhj Y`mdfcWYgg'ch\Yf'ghi XYbhg' Wcbhf]Vi hJcbg']b'fY`UhJcb'hc'h\Y]f'ck bz'UbX'hc'gY`ZXYhYfa]bY'U'Wf]nJWU``UbX'UbU`mhJWfYgYUfW\'dfUWhJWY''

5ggYgga Ybh

5ggYgga Ybh'HUg_g'

5ggYgga Ybh'tUg_	K Y][\h	8iY'8UHY	Gh XYbh@YUfb]b[Ci hWca Ygʻ5ggYggYX
9ggUm) \$1	\$&#\$(#&\$&\$`%%)-`DA	%Ž Ž(Ž) Ž+
; fci d'dfYgYbHJh]cb'	&) ı	&%#\$(#&\$&\$`\$&.\$\$`DA	&Ž Ž) Ž* Ž,
±bX]j]Xi U``DfcYWiFUh]cbU`Y`	&) ı	&, #\$(#&\$&\$`%%) - `DA	%ž&Ž Ž(Ž* Ž+

5ggYgga Ybh8YlU]`g

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GHJfh:XUhY.

@Yb[h\. &ž)\$\$ k cfXg

8YHJ]`g. '€X]j]Xi U`&ž) \$\$ k cfXg: YYXVUW_j j]U '@A G

 $\label{eq:higher} \begin{tabular}{ll} High \$

5ggYgga Ybh&; fci d dfYgYbhJhJcb

GhUfh XUhY.

@Yb[h\. '&\$'a]bi hYg

8YHJ]`g. `&\$!a]bi HYg: YYXVUW<u>'</u>j]U '@A G

Hi fb]hjb gYhjb[. H\]g]g bchU Hi fb]hjb Ugg][ba Ybh

5ggYgga Ybh' . ⇒x]j]Xi U`Dfc YWhF Uh]cbU`Y

GHJfh:XUhY.

@Yb[h\. *\%\\$\$\$!\%\&) \$ k cfXg

 $\label{thm:limit} \begin{tabular}{ll} Hi fb]h]b gYhh]b[. Hi]g Ugg][ba Ybh]g gi Va]hhYX hi fci [in Hi fb]h]b UbX ghi XYbhg Xc bchgYY Hi fb]h]b g]a] [in Hi]hmfYdcfhg" \\ \begin{tabular}{ll} Hi fb]h]b UbX ghi XYbhg Xc bchgYY Hi fb]h]b UbX ghi XYbhg Xc bchg Xc bchg$



FYgci fWyg

DfYgWf]VYX F Ygci fWYg

F Yei]fYX'UbX'Gi [[YghYX'F YUX]b[g'UfY``]ghYX'cb'A ccX`Y"'F YUX]b[g'a Um'W\Ub[Yžcf'bYk 'gi [[YghYX fYUX]b[g'UXXYXžgc'a U_Y'gi fY'hc'fY[i `Uf'm'W\YW_'A ccX`Y'Zcf'h\Y'a cghi d!hc!XUhY']bZcfa Uh]cb"

FYWca a YbXYX FYgci fW/g

: i ft\Yf`fYgci fWYg`WUb`VY`Zci bX`cb`A ccX`Y"

7ci fgY 9j U'i Uh]cb UbX 8Yj Y`cda Ybh

K Y`k Y`Wca Y`Wcbghfi Whj Y`ZYYXVUW_`-YgdYWjU``mik f]hhyb`ZYYXVUW_`flya U]`gž'bchyg`YhW'Lcb'h\Y`Wci fgY Wcbhybhzghfi Whi fy`UbX`UggYgga Ybha Yh\cXg`-'bchcb`mUhih\Y`WcbWi g]cb`cZh\Y`Wci fgYžVi h'U`gc`Ug`k Y`[c U`cb["'5hh\Y`YbX`cZh\Y`Wci fgY`k Y`k]```]bj]hy`mci 'hc`dfcj]XY`ZYYXVUW_`i g]b['I BGK fig`Wci fgY`Yl dYf]YbW' gi fj Ym'

A UbmUgdYWg`cZh\]g`Wci fgY`UfY`h\Y`fYgi`hcZ]bbcj Uh]j Y`ghi XYbhZYXVUW_`Z̄ca ˈdfYj]ci g`mYUfg": cf Yl Ua d`Yž'h\Y`bi a VYf`UbX`[YbfY`cZWUgY`ghi X]Yg`]b`h\]g`Wci fgY`\Ug`VYYb`W\Ub[YX`]b`fYgdcbgY`hc`ghi XYbh ZYXVUW_"`K]h\`h\Ya cj Y`hc`U`%\$!k YY_`hYUW\]b[`hYfa `]b`&\$%- žk Y`fYUffUb[YX`gca Y`WcbhYbhž'W\Ub[YX gca Y`UggYgga Ybhgž'UbX`]bhfcXi WYX`(!\ci f`gYa]bUfg"`H\]g`mYUfžk Y`\Uj Y`fYXi WYX`h\Y`bi a VYf`cZ UggYgga YbhhUg_g`Z̄ca 'Z̄ci f`hc`h\fYY"

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Hi fb]h]b Gi Va]gg]cb

Image: Ybwb: bh/f'u'dfcv'ya k \yb'uhh/a dh/b['hc'gi va]hmci f'ugg][ba Ybhh\fci [\ 'Hi fb]h/bž'd'YugY

HY'Yd\cbY'9| h/fbu''Gi ddcfhcb'-',) '' ' ' %cf'Ya U] 'h\Ya 'cb'Y! h/fbu'h/hgi ddcfh4 i bgk "YXi "Ui "'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY_XUmg'UbX'-.\$\$Ua '-') .\$\$da 'cb'k YY_YbXg'fl *) 'XUmg'U'mYUfŁ" Image: UfY'i bUV'Y'hc'gi va]hmci f'Ugg][ba YbhXi Y'hc'U'ZJi 'hk]h\ 'Hi fb]h/b'mci 'a UmUdd'mZcf'Ub'Y! h/bg]cbž'Vi hmci
a i ghfYhJ]b'mci f'h/w_Yhbi a VYf'Zfca '9! h/fbu''Gi ddcfhfU'cb['k]h\ 'Ubmch\Yf'YY] UbhXcW a YbhgŁ'hc
]bWi XY'Ug'Yj]XYbW'hc'gi ddcfhmci f'Y! h/bg]cb'Udd']WJh/cb" Image: Ya U]''9! h/fbu''Gi ddcfhmci 'k]'`
Ui hca Uh/WJ'mfYW'] Y'U'h/w_Yhbi a VYfž'Vi h]Zmci 'hY'Yd\cbY'mci 'k]'`'bYYX'hc'gdYW/ZWJ'`mUg_'Zcf'cbY''
Hi fb]h/b'U'gc'dfcj]XYg'i dXUh/g'cb'h\Y]f'gmghYa 'ghUh' g'cb'Hk]hYf"

; YbYfU``mž'UggYgga YbhiHug_g'a i ghVY'gi Va]hhYX'Y`YVWfcb]\W)``mj]U'Y]h\Yf'Hi fb]hjb'cf'U'A ccX`Y Ugg][ba Ybh'' =b']bgHJb\W'g'k \YfY'h\]g']g'bchdcgg]V`Yž']hk]``'VY'gHJhYX'cb'mci f'\\circ i fgY'g'A ccX`Y'g]hY'k]h\ U`hYfbUhj Y'gi Va]gg]cb'XYHJ]`g''

cfi]bZcfa Uh]cbicbi\ckihcigi Va]hUgg][ba Ybhgicbi]bYij]UiA ccXiY.i\hdg.#gh XYbh'i bgk "YXi "Ui #\ck! gi Va]hUgg][ba Ybha ccXiY

5 WUXYa | W < cbYghmUbX D`U[| Uf]ga

D`U[]Uf]ga i]gii g]b[ih\Yik cfXgicfi]XYUgicZch\YfgiUbXidfYgYbh]b[ih\Ya iUgimci fick b"i±iWUbihU_Yia Ubm Zcfa qžZfca iXYi]VYfUhYiW\YUh|b[ihciUVW]XYbhU`mWcdm]b[iZfca iUigci fWik]h\ci hiUW_bck iYX[Ya Ybh'i

IBGK '[fci dg'd'U[]Uf]ga ']bhc'h\Y'Zc``ck]b['WUhY[cf]Yg.

7 cdm]b[.'i g]b['h\Y'gUa Y'cf'j Yfmg]a]`Uf'k cfXg'hc'h\Y'cf][]bU`hYl h'cf']XYU'k]h\ci h'UW_bck `YX[]b['h\Y gci fW'cf'i g]b['ei chUh]cb'a Uf_g"'H\]g'U`gc'Udd`]Yg'hc']a U[Ygž'Ufh'UbX'XYg][b'dfcYWhgž'Ug'k Y``'Ug dfYgYbhUh]cbg'k\YfY'gca YcbY'dfYgYbhg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci h'WfYX]h'

⇒ Uddfcdf]UhY 'dUfUd\fUg]b[...7\Ub[]b['U'ZYk 'k cfXg'UbX'd\fUgYg'k \]'Y 'a cghmfYhJ]b]b['h\Y'cf][]bU` ghfi Wh fY 'UbX#cf'dfc[fYgg]cb'cZ]XYUg'cZh\Y'cf][]bU'žUbX']bZcfa Uh]cb'k]h\ci hUW_bck `YX[Ya Ybh'H\]g'U`gc Udd`]Yg']b'dfYgYbhUh]cbg'k \YfY'gca YcbY'dUfUd\fUgYg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci hWfYX]hUbX'hc d]YW]b['hc[Yh\Yf'ei chYg'UbX'dUfUd\fUgYg']bhc'U'bYk 'k \c`Yžk]h\ci hUddfcdf]UhY'fYZYfYbW]b["

7 c``i g]cb. 'k cf_]b['k]th 'cth Yfg'Vi hidUgg]b['cZZth Y 'k cf_'Ug'U'dYfgcb'g']bX]j]Xi U``k cf_"'7 c``i g]cb'U`gc]bWi XYg'dfcj]X]b['mci f'k cf_'thc'Ubcth Yf'ghi XYbh'VYZcfY'th Y'Xi Y'XUHYZcf'Zcf'th Y'di fdcgY'cZth Ya

- i bXYfgfUbX mci f'f][\hg'UbX'fYgdcbg]V]`]hJYg'Ug'U'gh XYbhUhl BGK
- VY`Uk UfY`cZd`U[]Uf]ga ž`Wcdmf][\hz'l BGK `Gh XYbh7cXY`cZ7cbXi WhUbX`5VWWdhUV`Y`l gY`cZ I BGK `=\forall H`F Ygci f\W\g`Dc`]\m\m
- VY 'Uk UfY 'cZh\Y 'ghUbXUfXg 'cZVY\Uj]ci f'Yl dYWYX 'cZYj YfmcbY ']b 'h\Y 'l BGK 'Wca a i b]hm
- `cWUHY 'gYfj]WYg'UbX ']bZcfa UH]cb'UVci hI BGK 'UbX'I BGK '@]VfUfm

Gca Y'cZh\YgY'UfYUg'k]``VY'ZJa]`]Uf'hc'mci ž'ch\Yfg'k]``VY'bYk"'; U]b]b['U'gc`]X'i bXYfgHJbX]b['cZU``h\YfY`UhYX'UgdYWng'cZ9@+69'k]``\Y`d'mci 'a U_Y'h\Y'a cghicZmci f'ghi X]Yg'UhI BGK "'\\hd.##qi VYWfi |XYg"`]VfUfm'i bqk "YXi "Ui #\`]qY#JVci hY`]qY

5 WUXYa]W±bZcfa Uhjcb

: cf'YggYbhJU`gh XYbh]bZcfa UhJcb'fY`UhJb['hc.